

## ASYMPTOTIC STABILITY FOR KÄHLER-RICCI SOLITONS

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ABSTRACT. Let  $X$  be a Fano manifold. We say that a hermitian metric  $\phi$  on  $-K_X$  with positive curvature  $\omega_\phi$  is a Kähler-Ricci soliton if it satisfies the equation  $\text{Ric}(\omega_\phi) - \omega_\phi = L_{V_{KS}}\omega_\phi$  for some holomorphic vector field  $V_{KS}$ . The candidate for a vector field  $V_{KS}$  is uniquely determined by the holomorphic structure of  $X$  up to conjugacy, hence depends only on the holomorphic structure of  $X$ . We introduce a sequence  $\{V_k\}$  of holomorphic vector fields which approximates  $V_{KS}$  and fits to the quantized settings. Moreover, we also discuss about the existence and convergence of the quantized Kähler-Ricci solitons attached to the sequence  $\{V_k\}$ .

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## 1. INTRODUCTION

Let  $X$  be an  $n$ -dimensional Fano manifold and  $PSH(X, -K_X)$  the set of (possibly singular) hermitian metrics  $\phi$  on the anti-canonical bundle  $-K_X$  with positive curvature current

$$\omega_\phi := \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi.$$

We regard  $\phi$  as a psh weight, i.e., a psh function on  $K_X \setminus \{0\text{-section}\}$  satisfying the log-homogeneity property (e.g., see [BB10] for more detail). Let  $\mathcal{H}(X, -K_X)$  be the subset of  $PSH(X, -K_X)$  consisting of all smooth psh weights. We say that a metric  $\phi \in \mathcal{H}(X, -K_X)$  is a Kähler-Ricci soliton if it satisfies the equation

$$\text{Ric}(\omega_\phi) - \omega_\phi = L_{V_{KS}}\omega_\phi$$

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for some holomorphic vector field  $V_{KS}$  (**Kähler-Ricci soliton vector field**), where  $L_{V_{KS}}$  denotes the Lie derivative with respect to  $V_{KS}$ . Kähler-Ricci solitons arise from the geometric analysis, such as Kähler-Ricci flow, and have been studied extensively for recent years. For instance, Chen-Wang [CW14] solved the Hamilton-Tian conjecture, which says that Kähler-Ricci flow on a Fano manifold always converges to a singular Kähler-Ricci soliton defined on a singular normal Fano variety in the sense of Cheeger-Gromov.

In this paper, we study the existence problem of Kähler-Ricci solitons. As shown in [TZ02], a necessary condition for the existence of a Kähler-Ricci soliton is the vanishing of **the modified Futaki invariant**: let  $\text{Aut}_0(X)$  be the identity component of the automorphism group of  $X$ . Since  $\text{Aut}_0(X)$  is a linear algebraic group [Fuj78], we obtain a semidirect decomposition

$$\text{Aut}_0(X) = \text{Aut}_r(X) \ltimes R_u,$$

where  $\text{Aut}_r(X)$  is a reductive algebraic subgroup (uniquely determined up to conjugacy of  $\text{Aut}_0(X)$ ), which is the complexification of a maximal compact subgroup  $K$ , and  $R_u$  is the unipotent radical. We often identify a holomorphic vector field  $V$  such that  $\text{Im}(V) \in \mathfrak{k} := \text{Lie}(K)$  with its imaginary part  $\xi_V := \text{Im}(V) \in \mathfrak{k}$ . Let  $T_c$  be a real torus defined as the identity component of the center of  $K$  and put  $\mathfrak{t}_c := \text{Lie}(T_c)$ . Then Tian-Zhu (cf. [TZ00], [TZ02]) showed that all Kähler-Ricci solitons are contained in the space of  $K$ -invariant smooth psh weights  $\mathcal{H}(X, -K_X)^K$  and  $\xi_{V_{KS}}$  is contained in  $\mathfrak{t}_c$ , which is uniquely determined as the minimizer of the following proper strictly convex function on  $\mathfrak{k}$ :

$$\mathcal{F}(V) := \int_X e^{m_\phi(\xi_V)} MA(\phi),$$

where  $\phi \in \mathcal{H}(X, -K_X)^K$ ,  $MA(\phi) := \frac{\omega_\phi^n}{c_1(X)^n}$  and  $m_\phi$  is the moment map with respect to  $\phi$ . The function  $\mathcal{F}$  is a holomorphic invariant, i.e., independent of a choice of  $\phi \in \mathcal{H}(X, -K_X)^K$  and its derivative at  $V$ :

$$\text{Fut}_V(W) := - \int_X m_\phi(\xi_W) e^{m_\phi(\xi_V)} MA(\phi)$$

is called the modified Futaki invariant.

Thereafter, Berman-Nyström [BN14] generalized the modified Futaki invariant for singular normal Fano varieties with a torus action, and introduced the notion of algebro-geometric stability, called K-polystability. They also showed that any Fano manifolds admitting a Kähler-Ricci soliton are K-polystable. The converse is also true at least in the case of Kähler-Einstein metrics (cf. [CDS15], [Tia15]), however, it is still open in general.

K-polystability is not only the notion of stability for the existence problem of Kähler-Ricci solitons. We study several functionals on the space of hermitian metrics and their asymptotics near the boundary. Then the notion of analytical K-polystability is defined as the strong properness (or coercivity) of the modified Ding functional. On the other hand, we also consider its analogue in the space  $\mathcal{H}_k$  of hermitian inner products on  $H^0(X, -kK_X)$  and study their asymptotics as

$k$  tends to infinity. Critical points of the quantized functionals are called balanced metrics. The existence of a balanced metric is closely related to the stability of the projective embedding of  $X$  (cf. [Don02], [Don09]). Berman-Nyström [BN14] showed that there exist a certain kind of balanced metrics, called **quantized Kähler-Ricci solitons** and this sequence of metrics converges to a Kähler-Ricci soliton as  $k$  tends to infinity under some strong assumptions.

The main purpose of this paper is to perturb the sequence of quantized Kähler-Ricci solitons and show their existence and convergence under weaker assumptions. First, we construct a sequence  $\{V_k\}$  of holomorphic vector fields which approximates  $V_{KS}$  and fits to the quantized settings. More concretely, this sequence is given as the following:

**Theorem 1.1.** *Let  $X$  be a Fano manifold and  $K$  be a maximal compact subgroup of  $\text{Aut}_r(X)$ . Then for sufficiently large  $k$ , there exists a holomorphic vector field  $V_k$  such that its imaginary part is contained in  $\mathfrak{t}_c$  and the corresponding quantized modified Futaki invariant at level  $k$  vanishes on  $\mathfrak{k}^\mathbb{C}$ . The vector field  $V_k$  is characterized as the unique minimizer of the quantization of the function  $\mathcal{F}|_{\mathfrak{t}_c}$  at level  $k$  and converges to  $V_{KS}$  as  $k \rightarrow \infty$  in the usual topology of the finite dimensional vector space  $\mathfrak{t}_c$ .*

Second, we introduce the quantized Kähler-Ricci solitons attached to this sequence and show that:

**Theorem 1.2.** *Assume that  $(X, V_{KS})$  is strongly analytically  $K$ -polystable (i.e., the corresponding modified Ding functional is coercive modulo  $\text{Aut}_0(X, V_{KS})$ ), then there exists a quantized Kähler-Ricci soliton attached to  $V_k$  if  $k$  is sufficiently large, which is unique modulo the action of  $\text{Aut}_0(X, V_{KS})$  and as  $k \rightarrow \infty$ , the corresponding Bergman metrics on  $X$  converge weakly, modulo automorphisms, to a Kähler-Ricci soliton on  $(X, V_{KS})$ .*

As a corollary, we have the following:

**Corollary 1.1.** *Assume that  $X$  is strongly analytically  $K$ -polystable (i.e., the Ding functional is coercive modulo  $\text{Aut}_0(X)$ ), then there exists a quantized Kähler-Einstein metric attached to  $V_k \rightarrow 0$  if  $k$  is sufficiently large, which is unique modulo the action of  $\text{Aut}_0(X)$  and as  $k \rightarrow \infty$ , the corresponding Bergman metrics on  $X$  converge weakly, modulo automorphisms, to a Kähler-Einstein metric on  $X$ .*

The crucial point is that in our results, we need not to assume that the vanishing of all the higher order (modified) Futaki invariants, which is, in the case of  $V_{KS} \equiv 0$ , an obstruction to the asymptotic Chow semi-stability (cf. [Fut04]). Thus we can apply our results to even asymptotically Chow unstable Fano manifolds like [OSY12].

Now we describe the content of this paper. In the next section, we review the basic of several functionals on the space of smooth psh weights  $\mathcal{H}(X, -K_X)$ . The standard reference for this section is [BN14]. However, in order to prove the convergence of quantized metrics in Theorem 1.2, we need to extend functionals for (possibly singular) psh weights, which requires a lot of knowledge about the pluripotential theory (e.g. [BBGZ13], [BE10], [BN14]). So the reader should see these references as needed.

In Section 3, we introduce the quantization of the modified Futaki invariant  $\text{Fut}_{V,k}$  and show that the functional  $\text{Fut}_{V,k}$  restricted to  $\mathfrak{t}_c$  is strictly proper convex, and hence has a unique minimizer  $V_k$ . This observation and the quantization formula (cf. Lemma 3.2) yield Theorem 1.1. Then we review the basic properties of the quantized functionals on  $\mathcal{H}_k$  studied in [BN14] and prove Theorem 1.2. The heart of the proof of Theorem 1.2 consists of mainly two ideas:

- (1) While Berman-Nyström considered the torus  $T_{KS}$  generated by the Kähler-Ricci soliton vector field  $V_{KS}$ , we consider the identity component of the center  $T_c(\supset T_{KS})$  and the space of  $T_c$ -invariant hermitian metrics  $\mathcal{H}(X, -K_X)^{T_c}$ . Actually, this setting seems to be natural since all of  $\xi_{V_k}$  lie in its Lie algebra  $\mathfrak{t}_c$  by Theorem 1.1.
- (2) The condition  $\text{Fut}_{V_{KS}} \equiv 0$  (resp.  $\text{Fut}_{V_k,k} \equiv 0$ ) leads to the  $\text{Aut}_0(X, V_{KS})$ -invariance of the functional  $\mathcal{D}_{g_{V_{KS}}}$  (resp.  $\mathcal{D}_{g_{V_k}}^{(k)}$ ). Hence the problem can be reduced to estimate the difference  $\mathcal{D}_{g_{V_{KS}}}^{(k)} - \mathcal{D}_{g_{V_k}}^{(k)}$ , which is linear along geodesics. On the other hand, the standard exhaustion function  $J^{(k)}$  has at least linear growth along geodesics. Therefore the absolute of  $\mathcal{D}_{g_{V_{KS}}}^{(k)} - \mathcal{D}_{g_{V_k}}^{(k)}$  is bounded above by an affine function  $\epsilon_k J^{(k)} + \epsilon'_k$  of  $J^{(k)}$  with some positive numbers  $\epsilon_k \rightarrow 0$  and  $\epsilon'_k \rightarrow 0$ . This leads to the coercivity of  $\mathcal{D}_{g_{V_k}}^{(k)}$  and therefore the existence of the quantized Kähler-Ricci soliton attached to  $V_k$ .

Finally, we mention a result for extremal Kähler metrics proved by Mabuchi [Mab09], which says that any polarized manifolds admitting an extremal Kähler metric are asymptotically Chow stable relative to an algebraic torus. It seems that such a stability for Kähler-Ricci solitons has never been discussed, but it is known that relative Chow stability leads to the existence of “polybalanced metrics” (cf. [Mab11]). That is why Theorem 1.2 can be seen as an analogue of Mabuchi’s result indirectly.

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## 2. FUNCTIONALS ON $\mathcal{H}(X, -K_X)^T$ AND ANALYTIC K-POLYSTABILITY

Let  $X$  and  $V_{KS}$  be as in Section 1. Then we see that  $V_{KS}$  generates a torus action on  $(X, -K_X)$ :

**Lemma 2.1** ([BN14], Lemma 2.13). *There is a real torus  $T_{KS} \subset \text{Aut}_r(X)$  which acts on  $(X, -K_X)$  such that the imaginary part of  $V_{KS}$  can be identified with an element  $\xi_{V_{KS}} \in \mathfrak{t}_{KS} := \text{Lie}(T_{KS})$  and  $\mathcal{H}(X, -K_X)^{V_{KS}} = \mathcal{H}(X, -K_X)^{T_{KS}}$ .*

Let  $T \subset \text{Aut}_r(X)$  be an  $m$ -dimensional real torus acting on  $(X, -K_X)$ . For  $\phi \in \mathcal{H}(X, -K_X)^T$ , we denote the moment map with respect to  $\phi$  by

$$m_\phi: X \rightarrow m_\phi(X) =: P \subset \mathfrak{t}^* \simeq \mathbb{R}^m,$$

where we identify  $\mathfrak{t}^* \simeq \mathbb{R}^m$  using an inner product on  $\mathfrak{t}$ . The image  $P$  is a compact convex polytope of dimension  $m$ , characterized as the support of the Duistermaat-Heckman measure

$$\nu^T := (m_\phi)_* MA(\phi),$$

which is independent of a choice of  $\phi \in \mathcal{H}(X, -K_X)^T$  (cf. Proposition 3.1).

Now we recall several functionals on  $\mathcal{H}(X, -K_X)^T$  which play a central role in the study of Kähler-Ricci solitons. Let  $\phi_0 \in \mathcal{H}(X, -K_X)^T$  be a reference metric and  $g$  a positive continuous function on the moment polytope  $P$ . We normalize  $g$  so that  $g\nu^T$  is a probability measure on  $P$ . Following [BN14, Section 2.4 and 2.6], we define the  $g$ -Monge-Ampère energy by the formula

$$d\mathcal{E}_g|_\phi(\dot{\phi}) = \int_X \dot{\phi} MA_g(\phi), \quad \mathcal{E}_g(\phi_0) = 0,$$

where  $MA_g(\phi) := g(m_\phi)MA(\phi)$  denotes the  $g$ -Monge-Ampère measure. Then the functional  $\mathcal{E}_g$  satisfies the scaling property:

$$\mathcal{E}_g(\phi + c) = \mathcal{E}_g(\phi) + c$$

for all  $\phi \in \mathcal{H}(X, -K_X)^T$  and  $c \in \mathbb{R}$ . Let  $\mu_\phi$  be a measure on  $X$  given by the local expression

$$\mu_\phi = e^{-\phi_U} (\sqrt{-1})^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n,$$

where  $(U; z_1, \dots, z_n)$  denotes a holomorphic local coordinates and  $\phi_U := \log \left| \frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n} \right|_\phi^2$ .

We define functionals  $J_g$  and  $\mathcal{D}_g$  by

$$J_g(\phi) = -\mathcal{E}_g(\phi) + \mathcal{L}_{\mu_0}(\phi), \quad \mathcal{L}_{\mu_0}(\phi) := \int_X (\phi - \phi_0) d\mu_0,$$

$$\mathcal{D}_g(\phi) = -\mathcal{E}_g(\phi) + \mathcal{L}(\phi), \quad \mathcal{L}(\phi) := -\log \int_X d\mu_\phi,$$

where  $\mu_0 := MA(\phi_0)$  is a fixed probability measure on  $X$ . One can easily see that  $J_g$  and  $\mathcal{D}_g$  are invariant under scaling of metrics, i.e., functionals on  $\mathcal{H}(X, -K_X)^T/\mathbb{R}$ . When  $g \equiv 1$ , we simply write these functionals by  $\mathcal{E}$ ,  $J$  and  $\mathcal{D}$  respectively.

*Remark 2.1.* We can extend these functionals to the space  $\mathcal{E}^1(X, -K_X)^T$  of all  $T$ -invariant (possibly singular) psh weights with finite energy (cf. [BN14, Section 2 and Section 3]). Moreover, the functional  $J$  defines an exhaustion function on  $\mathcal{E}^1(X, -K_X)/\mathbb{R}$  in the sense that each level set of  $J$  is compact in  $L^1$ -topology (cf. [BBGZ13, Lemma 3.3]).

Next we set  $T = T_{KS}$  and  $g = g_{V_{KS}} := \exp(\langle \xi_{V_{KS}}, \cdot \rangle)$ . Then the corresponding functional  $\mathcal{D}_g = \mathcal{D}_{g_{V_{KS}}}$  is called the modified Ding functional. By [BN14, Lemma 3.4], we have

$$\frac{d}{dt} \mathcal{D}_{g_{V_{KS}}}(\exp(tW)^*\phi) = \text{Fut}_{V_{KS}}(W).$$

Moreover, critical points of  $\mathcal{D}_{g_{V_{KS}}}$  are Kähler-Ricci solitons with respect to  $V_{KS}$ .

**Definition 2.1** ([BN14], Section 3.6). We say that a pair  $(X, V_{KS})$  is strongly analytically K-polystable if the modified Ding functional  $\mathcal{D}_{gV_{KS}}$  is coercive modulo  $\text{Aut}_0(X, V_{KS})$ , i.e.,

$$\mathcal{D}_{gV_{KS}}(\phi) \geq \delta \inf_{F \in \text{Aut}_0(X, V_{KS})} J(F^*\phi) - C, \quad \phi \in \mathcal{H}(X, -K_X)^{T_{KS}}$$

for some positive constants  $\delta$  and  $C$ , where  $\text{Aut}_0(X, V_{KS})$  be a subgroup of  $\text{Aut}_0(X)$  consisting of elements which commute with the action generated by  $V_{KS}$ .

**Theorem 2.1** ([BN14], Theorem 3.11). *If a pair  $(X, V_{KS})$  is strongly analytically K-polystable, then  $(X, V_{KS})$  admits a Kähler-Ricci soliton (as a unique minimizer of the modified Ding functional up to the action of  $\text{Aut}_0(X, V_{KS})$ ).*

### 3. THE QUANTIZED SETTING

**3.1. The quantization of  $\mathcal{F}$ -functional, the modified Futaki invariant and the Duistermaat-Heckman measure.** Let  $X$  and  $T$  be as in Section 2.

**Lemma 3.1.** *Assume that  $m := \dim T$  is greater than 1. Then the polytope  $P$  contains the origin in its interior  $\text{int}(P)$ .*

*Proof.* Since the lift to  $-K_X$  is canonical, we have an equation

$$-\Delta_\partial m_\phi(\xi_V) + m_\phi(\xi_V) + V(\kappa_\phi) = 0$$

for all  $\xi_V \in \mathfrak{t}$ , where  $\kappa_\phi$  is the function defined by

$$\text{Ric}(\omega_\phi) - \omega_\phi = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \kappa_\phi.$$

Integrating by parts, we find that

$$(3.1) \quad \int_X m_\phi(\xi_V) e^{\kappa_\phi} MA(\phi) = 0.$$

Since  $m \geq 1$ ,  $m_\phi$  is not a constant. Thus the equation (3.1) implies that for any  $\xi_V$ , an inequality  $m_\phi(\xi_V) > 0$  holds on some nonempty open subset of  $X$ . Now we assume that  $0 \notin \text{int}(P)$ , then we can choose an element  $\xi \in \mathbb{R}^m$  so that  $H_\xi \cap \text{int}(P) = \emptyset$ , where  $H_\xi$  a hyperplane which is orthogonal to  $\xi$  and contains the origin. Then either of  $m_\phi(\pm \xi_V)$  is semi-negative on  $X$ . This is a contradiction.  $\square$

We define the functions  $\mathcal{F}_k$  and  $\text{Fut}_{V,k}$  as

$$(3.2) \quad \mathcal{F}_k(W) := k \text{Trace}(e^{W/k})|_{H^0(X, -kK_X)},$$

$$(3.3) \quad \text{Fut}_{V,k}(W) := - \left. \frac{d}{dt} \right|_{t=0} \mathcal{F}_k(V + tW).$$

We set

$$N_k := \dim H^0(X, -kK_X),$$

then these functions give the quantization of  $\mathcal{F}$  and  $\text{Fut}_V$ :

**Lemma 3.2** ([BN14], Proposition 4.7 or [Tak14], Proposition 2.8). *Let  $V$  be a holomorphic vector field generating a torus action and  $W$  a holomorphic vector field generating a  $\mathbb{C}^*$ -action and commuting with  $V$ . Then we have identities*

$$\mathcal{F}(V) = \lim_{k \rightarrow \infty} \frac{\mathcal{F}_k(V)}{kN_k},$$

$$\text{Fut}_V(W) = \lim_{k \rightarrow \infty} \frac{1}{kN_k} \text{Fut}_{V,k}(W).$$

If we apply the equivariant Riemann-Roch formula to  $\text{Fut}_{V,k}(W)$ , we have an expansion

$$\text{Fut}_{V,k}(W) = \text{Fut}_V^{(0)}(W)k^{n+1} + \text{Fut}_V^{(1)}(W)k^n + \cdots,$$

where  $\text{Fut}_V^{(i)}(W)$  is the  $i$ -th order modified Futaki invariant introduced in [BN14, Section 4.4].

**Lemma 3.3.** *The function  $\mathcal{F}_k|_{\mathfrak{t}_c}$  is a proper strictly convex function if  $k$  is sufficiently large.*

*Proof.* We use the following proposition:

**Proposition 3.1** ([BN14], Proposition 4.1). *Let  $P_k := \{\lambda_i^{(k)}\} \subset \mathbb{Z}^m$  be the set of all weights for the action of the complexified torus  $T^{\mathbb{C}}$  on  $H^0(X, -kK_X)$ , i.e., there is a decomposition*

$$H^0(X, -kK_X) = \bigoplus_{\lambda_i^{(k)} \in P_k} E_{\lambda_i^{(k)}}.$$

*Then the spectral measure:*

$$\nu_k := \frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{\lambda_i^{(k)}/k}$$

*supported on  $P_k/k$  converges to the Duistermaat-Heckman measure  $\nu^T$  weakly as  $k \rightarrow \infty$ , where  $\delta_{\lambda_i^{(k)}/k}$  denotes the Dirac measure at  $\lambda_i^{(k)}/k$ . In particular,  $\nu^T$  does not depend on a choice of  $\phi \in \mathcal{H}(X, -K_X)^T$ .*

For any  $\xi_V, \xi_W (\neq 0) \in \mathfrak{t}_c$ , the functional  $\mathcal{F}_k$  along the line  $\xi_V + t\xi_W$  ( $t \in \mathbb{R}$ ) can be written as the form

$$\mathcal{F}_k(V + tW) = k \sum_{i=1}^{N_k} \exp(v_i^{(k)}/k + tw_i^{(k)}/k),$$

where  $v_i^{(k)} := \langle \xi_V, \lambda_i^{(k)} \rangle$  and  $w_i^{(k)} := \langle \xi_W, \lambda_i^{(k)} \rangle$  are joint eigenvalues of the commuting action generated by  $\text{Re}(V)$  and  $\text{Re}(W)$ . Then Proposition 3.1 implies that the functional  $\mathcal{F}_k(V + tW)$  of  $t$  is strictly convex for any  $\xi_V, \xi_W \in \mathfrak{t}_c$  if  $k$  is sufficiently large and hence  $\mathcal{F}_k$  is strictly convex.

In order to prove the properness, let  $\{\xi_{W_j}\} \subset \mathfrak{t}_c \simeq \mathbb{R}^m$  be any sequence such that  $|\xi_{W_j}| \rightarrow \infty$  as  $j \rightarrow \infty$ . For  $\epsilon > 0$ , let  $P_\epsilon$  be the interior compact convex polytope with faces parallel to those of  $P$  separated by distance  $\epsilon$ . By Lemma 3.1, we can choose  $\epsilon > 0$  so that  $\text{int}(P_\epsilon)$  contains the origin. Then Proposition 3.1 implies that

there exists  $k_0$  such that for all  $k \geq k_0$  and  $\xi_W \in \mathbb{R}^m$ , there exists an eigenvalue  $\lambda_i^{(k)}$  satisfying

$$\begin{aligned} \lambda_i^{(k)}/k &\in P - P_\epsilon, \\ \cos(\text{angle}(\xi_W, \lambda_i^{(k)})) &\geq 1 - \epsilon. \end{aligned}$$

For each  $\xi_{W_j}$ , we choose the eigenvalue  $\lambda_{j,i(j)}^{(k)}$  satisfying the above condition. Then we obtain

$$w_{j,i(j)}^{(k)} := \langle \lambda_{j,i(j)}^{(k)}, \xi_{W_j} \rangle \geq k |\xi_{W_j}| \cdot \inf_{\xi \in \partial P_\epsilon} |\xi| \cdot (1 - \epsilon) \rightarrow \infty$$

as  $j \rightarrow \infty$ . Hence we have

$$\mathcal{F}_k(W_j) = k \sum_{i=1}^{N_k} \exp(w_{j,i}^{(k)}/k) \geq k \exp(w_{j,i(j)}^{(k)}/k) \rightarrow \infty$$

as  $j \rightarrow \infty$ . This completes the proof of Lemma 3.3.  $\square$

Let  $V_k$  be the unique minimizer of  $\mathcal{F}_k|_{\mathfrak{t}_c}$ .

*The proof of Theorem 1.1.* By Lemmas 3.2 and 3.3, we find that the unique minimizer  $V_k$  converges to the unique minimizer of  $\mathcal{F}$ , i.e., the Kähler-Ricci soliton vector field as  $k \rightarrow \infty$ . Since  $V_k \in \mathfrak{t}_c^\mathbb{C}$  and  $\mathcal{F}_k$  is adjoint invariant (so is Trace in the defining equation (3.2) of  $\mathcal{F}_k$ ), we have

$$\text{Fut}_{V_k,k}(\text{Ad}_F W) = - \left. \frac{d}{dt} \right|_{t=0} \mathcal{F}_k(V_k + t \text{Ad}_F W) = - \left. \frac{d}{dt} \right|_{t=0} \mathcal{F}_k(V_k + tW) = \text{Fut}_{V_k,k}(W)$$

for any  $F \in \text{Aut}_r(X)$  and  $W \in \mathfrak{k}^\mathbb{C}$ . In particular, set  $F_s := \exp(sV)$  ( $V \in \mathfrak{k}^\mathbb{C}$ ) and differentiating at  $s = 0$  yields

$$\text{Fut}_{V_k,k}([V, W]) = 0.$$

Moreover, the formula  $\mathfrak{k}^\mathbb{C} = \mathfrak{t}_c^\mathbb{C} \oplus [\mathfrak{k}^\mathbb{C}, \mathfrak{k}^\mathbb{C}]$  yields that  $\text{Fut}_{V_k,k}$  vanishes on the entire space  $\mathfrak{k}^\mathbb{C}$ . This completes the proof.  $\square$

**3.2. The  $g$ -Bergman measure and quantized functionals.** Let  $X$  be a Fano manifold,  $\mu$  a  $T$ -invariant measure, and  $\mathcal{H}_k$  the space of hermitian inner products on  $H^0(X, -kK_X)$ . We define two operators:

$$\begin{aligned} \text{Hilb}_{k,\mu,g}: \mathcal{H}(X, -K_X)^T &\rightarrow \mathcal{H}_k^T, \\ FS_k: \mathcal{H}_k^T &\rightarrow \mathcal{H}(X, -K_X)^T \end{aligned}$$

by the formula:

$$(3.4) \quad \|s_i^{(k)}\|_{\text{Hilb}_{k,\mu,g}(\phi)}^2 := g(\lambda_i^{(k)}/k)^{-1} \int_X |s_i^{(k)}|^2 e^{-k\phi} d\mu \quad \text{for } s_i^{(k)} \in E_{\lambda_i^{(k)}},$$

$$(3.5) \quad FS_k(H) := \frac{1}{k} \log \left( \frac{1}{N_k} \sum_{i=1}^{N_k} |s_i|^2 \right),$$

where we denote an  $H$ -orthonormal basis by  $\{s_i\}$  and extend  $\text{Hilb}_{k,\mu,g}$  to a hermitian inner product on the space  $H^0(X, -kK_X)$  by requiring that the different subspace



$E_{\lambda_i^{(k)}}$  are orthogonal to each other. We note that the map  $FS_k$  is independent of a choice of an  $H$ -orthonormal basis  $\{s_i\}$ . Actually,  $FS_k(H)$  is just the pull-back of the Fubini-Study metric with respect to  $H$ . In the case when  $\mu = \mu_\phi$ , we drop the explicit dependence of  $\mu$  from the notation and simply write

$$Hilb_{k,g}(\phi) := Hilb_{k,\mu_\phi,g}(\phi).$$

Let  $H_0 := Hilb_{k,\mu_0}(\phi_0) \in \mathcal{H}_k^{T_c}$  ( $\phi_0 \in \mathcal{H}(X, -K_X)^{T_c}$ ) be a reference metric. We normalize  $g$  so that  $g\nu_k$  is a probability measure on  $P$ . We define the quantization of the functional  $\mathcal{E}_g$  as

$$(3.6) \quad \mathcal{E}_g^{(k)}(H) = \sum_{\lambda_i^{(k)} \in P_k} g(\lambda_i^{(k)}/k) \mathcal{E}_{E_{\lambda_i^{(k)}}}^{(k)}(H), \quad \mathcal{E}_{E_{\lambda_i^{(k)}}}^{(k)}(H) = -\frac{1}{kN_k} \log \det H|_{E_{\lambda_i^{(k)}}},$$

where we compute the determinant in reference to the metric  $H_0$ . We have an isomorphism

$$\mathcal{H}_k \simeq GL(N_k, \mathbb{C})/U(N_k)$$

with respect to  $H_0$ , which implies that  $\mathcal{H}_k$  is a Riemannian symmetric space and therefore geodesics are given as the decomposition of the exponential map and the projection  $GL(N_k, \mathbb{C}) \rightarrow GL(N_k, \mathbb{C})/U(N_k)$ . Let  $s_i^{(k)} \in E_{\lambda_i^{(k)}}$  be an  $H_0$ -orthonormal and  $H_t$ -orthogonal basis. Then any geodesic can be represented by  $H_t(s_i^{(k)}, s_i^{(k)}) = e^{-\mu_i^{(k)}t} H_0(s_i^{(k)}, s_i^{(k)})$  for some  $\mu_i^{(k)} \in \mathbb{R}$ . Thus we have

$$\frac{d}{dt} \mathcal{E}_g^{(k)}(H_t) = \frac{1}{kN_k} \sum_{i=1}^{N_k} g(\lambda_i^{(k)}/k) \mu_i^{(k)}.$$

Hence the functional  $\mathcal{E}_g^{(k)}$  has linear growth along geodesics. We define the quantization of the functionals  $J_g, \mathcal{D}_g$  as follows:

$$(3.7) \quad J_g^{(k)}(H) := -\mathcal{E}_g^{(k)}(H) + \mathcal{L}_{\mu_0}(FS_k(H)),$$

$$(3.8) \quad \mathcal{D}_g^{(k)}(H) := -\mathcal{E}_g^{(k)}(H) + \mathcal{L}(FS_k(H)).$$

These functionals are invariant under scaling of metrics and descend to functionals on the space  $\mathcal{H}_k^{T_c}/\mathbb{R}$ . When  $g \equiv 1$ , we simply write these functionals by  $\mathcal{E}^{(k)}, J^{(k)}$  and  $\mathcal{D}^{(k)}$  respectively.

Now we will explain that the quantized functionals  $\mathcal{E}_g^{(k)}, J_g^{(k)}$  and  $\mathcal{D}_g^{(k)}$  are, to the letter, the quantization of the corresponding functionals on  $\mathcal{H}(X, -K_X)^T$  respectively. We start with mentioning the  $g$ -Bergman function:

$$(3.9) \quad \rho_{k,\mu_0,g}(\phi) := \sum_{\lambda_i^{(k)} \in P_k} g(\lambda_i^{(k)}/k) \rho_{k,\mu_0}(\phi)$$

and  $g$ -Bergman measure

$$(3.10) \quad \beta_{k,\mu_0,g}(\phi) := \frac{1}{N_k} \rho_{k,\mu_0,g}(\phi) \cdot \mu_0,$$

where  $\rho_{k,\mu_0,g}(\phi)$  is the ordinary Bergman function of the subspace  $E_{\lambda_i^{(k)}}$ . We use the following convergence of measures:

**Proposition 3.2** ([BN14], Proposition 4.4). *Assume that  $g$  is smooth, then for any  $\phi \in \mathcal{H}(X, -K_X)^T$ , we have the uniform convergence*

$$\beta_{k,\mu_0,g}(\phi) \rightarrow MA_g(\phi).$$

Now we are ready to prove the quantization formula (cf. [BN14, Proposition 4.5]).

**Proposition 3.3.** *The following pointwise convergence holds as  $k \rightarrow \infty$ :*

$$\mathcal{E}_g^{(k)}(Hilb_{k,\mu_0}(\phi)) \rightarrow \mathcal{E}_g(\phi),$$

$$J_g^{(k)}(Hilb_{k,\mu_0}(\phi)) \rightarrow J_g(\phi),$$

$$\mathcal{D}_g^{(k)}(Hilb_{k,\mu_0}(\phi)) \rightarrow \mathcal{D}_g(\phi).$$

*Proof.* The direct computation yields that

$$\begin{aligned} \mathcal{E}_g^{(k)}(Hilb_{k,\mu_0}(\phi)) &= \int_0^1 \left( \frac{d}{dt} \mathcal{E}_g^{(k)}(Hilb_{k,\mu_0}(t\phi + (1-t)\phi_0)) \right) dt \\ &\quad (\text{because } \mathcal{E}_g^{(k)}(Hilb_{k,\mu_0}(\phi_0)) = 0) \\ &= \int_0^1 \int_X (\phi - \phi_0) \beta_{k,\mu_0,g}(t\phi + (1-t)\phi_0) \\ &\quad (\text{by [BN14, Proposition 4.5]}). \end{aligned}$$

Combining with Proposition 3.2, we have

$$\mathcal{E}_g^{(k)}(Hilb_{k,\mu_0}(\phi)) \rightarrow \int_0^1 \int_X (\phi - \phi_0) MA_g(t\phi + (1-t)\phi_0) = \mathcal{E}_g(\phi).$$

Moreover, by definition,

$$(3.11) \quad FS_k \circ Hilb_{k,\mu_0,g}(\phi) - \phi = \frac{1}{k} \log \left( \frac{1}{N_k} \rho_{k,\mu_0,g}(\phi) \right).$$

Thus using Proposition 3.2 again, we have a uniform convergence

$$FS_k \circ Hilb_{k,\mu_0,g}(\phi) \rightarrow \phi.$$

The last two parts follow from the defining equations (3.7), (3.8) and pointwise convergence  $\mathcal{E}_g^{(k)} \circ Hilb_{k,\mu_0} \rightarrow \mathcal{E}_g$ .  $\square$

**3.3. Modification of quantized Kähler-Ricci solitons.** We adopt the same notation as in Section 3.2. We set  $T = T_{KS}$ .

**Definition 3.1** ([BN14], Section 4). We say that a metric  $H_k \in \mathcal{H}_k^{T_{KS}}$  is a quantized Kähler-Ricci soliton if it satisfies the equation

$$Hilb_{k,g_{V_{KS}}} \circ FS_k(H_k) = H_k.$$

Berman-Nyström showed the following:

**Theorem 3.1** ([BN14], Theorem 1.7). *Assume that  $(X, V_{KS})$  is strongly analytically  $K$ -polystable (i.e., the corresponding modified Ding functional is coercive modulo  $\text{Aut}_0(X, V_{KS})$ ) and all the higher order modified Futaki invariants of  $(X, V_{KS})$  vanish, then there exists a quantized Kähler-Ricci soliton if  $k$  is sufficiently large, which is unique modulo the action of  $\text{Aut}_0(X, V_{KS})$  and as  $k \rightarrow \infty$ , the corresponding Bergman metrics on  $X$  converge weakly, modulo automorphisms, to a Kähler-Ricci soliton on  $(X, V_{KS})$ .*

We want to weaken the assumption in the above theorem. For this, we set  $T = T_c$  and introduce a slight modification of the notion of quantized Kähler-Ricci solitons:

**Definition 3.2.** Let  $\{V_k\}$  be a sequence of holomorphic vector fields constructed in Section 3.1. We say that a metric  $H_k \in \mathcal{H}_k^{T_c}$  is a quantized Kähler-Ricci soliton attached to  $V_k$  if it satisfies the equation

$$\text{Hilb}_{k, g_{V_k}} \circ FS_k(H_k) = H_k.$$

Then the quantized Kähler-Ricci soliton attached to  $V_k$  are characterized as critical points of the quantization of the modified Ding functional  $\mathcal{D}_{g_{V_k}}^{(k)}$ . Moreover, by [BN14, Proposition 4.7], we have

$$\mathcal{D}_{g_{V_k}}^{(k)}(\exp(tW)^* H_0) = \frac{\text{Fut}_{V_k, k}(W)}{kN_k}.$$

*Proof of the Theorem 1.2.* This proof is mostly based on the original proof given by Berman-Nyström. The reader should refer to [BN14, Theorem 1.7].

The coercivity of  $\mathcal{D}_{g_{V_{KS}}}$  implies that the equation

$$\mathcal{D}_{g_{V_{KS}}}(FS_k(H)) \geq \delta J(FS_k(F^*H)) - C$$

holds for some  $F \in \text{Aut}_0(X, V_{KS})$ , where we note that two operations  $FS_k$  and  $F^*$  are commutative. Then the LHS can be written as

$$\begin{aligned} \mathcal{D}_{g_{V_{KS}}}(FS_k(H)) &= \mathcal{D}_{g_{V_{KS}}}(FS_k(F^*H)) \quad (\text{because } \text{Fut}_{V_{KS}} \equiv 0) \\ &= J_{g_{V_{KS}}}(FS_k(F^*H)) + (\mathcal{L} - \mathcal{L}_{\mu_0})(FS_k(F^*H)). \end{aligned}$$

On the other hand, since  $g_{V_{KS}}$  is bounded, we obtain

$$\delta J(FS_k(F^*H)) - C \geq \delta' J_{g_{V_{KS}}}(FS_k(F^*H)) - C$$

for sufficiently small  $\delta' > 0$  depending only on  $g_{V_{KS}}$ . Thus we obtain

$$(3.12) \quad J_{g_{V_{KS}}}(FS_k(F^*H))(1 - \delta') + (\mathcal{L} - \mathcal{L}_{\mu_0})(FS_k(F^*H)) \geq -C.$$

Now we use the following lemma, which compares the two functionals  $J_{g_{V_{KS}}} \circ FS_k$  and  $J_{g_{V_{KS}}}^{(k)}$ :

**Lemma 3.4** ([BN14], Lemma 4.10). *There exists a sequence  $\delta_k \rightarrow 0$  of positive numbers such that*

$$J_{g_{V_{KS}}}(FS_k(H)) \leq (1 + \delta_k) J_{g_{V_{KS}}}^{(k)}(H) + \delta_k.$$

Hence if we take  $k$  sufficiently large so that  $(1 + \delta_k)(1 - \delta') \leq 1 - \frac{\delta'}{2}$  and  $\delta_k(1 - \delta') \leq C$  hold, we have

$$(3.13) \quad J_{g_{V_{KS}}}(FS_k(F^*H))(1 - \delta') \leq J_{g_{V_{KS}}}^{(k)}(F^*H) \left(1 - \frac{\delta'}{2}\right) + C.$$

Thus we obtain

$$\begin{aligned} \mathcal{D}_{g_{V_{KS}}}^{(k)}(F^*H) &= J_{g_{V_{KS}}}^{(k)}(F^*H) + (\mathcal{L} - \mathcal{L}_{\mu_0})(FS_k(F^*H)) \\ &\geq \frac{\delta'}{2} J_{g_{V_{KS}}}^{(k)}(F^*H) - 2C \quad (\text{by (3.12) and (3.13)}) \\ &\geq \frac{\delta''}{2} J^{(k)}(F^*H) - 2C \quad (\text{because } g_{V_{KS}} \text{ is bounded}). \end{aligned}$$

Now we consider the difference of the two modified Ding functionals:

$$\mathcal{D}_{g_{V_{KS}}}^{(k)} - \mathcal{D}_{g_{V_k}}^{(k)} = -\mathcal{E}_{g_{V_{KS}}}^{(k)} + \mathcal{E}_{g_{V_k}}^{(k)},$$

which has linear growth along geodesics explained above. On the other hand, the functional  $J^{(k)}$  is an exhaustion function on  $\mathcal{H}_k^{T_c}/\mathbb{R}$  and has at least linear growth along geodesics (cf. [Don09, Proposition 3] or [BBGZ13, Lemma 7.6]). The following Lemma was inspired by these observations:

**Lemma 3.5.** *The inequality*

$$(3.14) \quad -\epsilon_k J^{(k)} - \epsilon'_k \leq \mathcal{D}_{g_{V_{KS}}}^{(k)} - \mathcal{D}_{g_{V_k}}^{(k)} \leq \epsilon_k J^{(k)} + \epsilon'_k$$

*holds for some sequences of positive numbers  $\epsilon_k \rightarrow 0$  and  $\epsilon'_k \rightarrow 0$ .*

*Proof.* We set  $\epsilon_k := \sup_P |g_{V_{KS}} - g_{V_k}| + 2^{-k}$  and define the functional  $\mathcal{E}_{\epsilon_k + g_{V_{KS}} - g_{V_k}}^{(k)} := \epsilon_k \mathcal{E}^{(k)} + \mathcal{E}_{g_{V_{KS}}}^{(k)} - \mathcal{E}_{g_{V_k}}^{(k)}$ . Then we have  $\epsilon_k \rightarrow 0$  since  $V_k \rightarrow V_{KS}$  and  $g_{V_k} \rightarrow g_{V_{KS}}$  uniformly on  $P$ . By scaling invariance of (3.14), we may assume that  $H$  is normalized by  $\mathcal{E}_{\epsilon_k + g_{V_{KS}} - g_{V_k}}^{(k)}(H) = 0$ . Now we consider a (non-trivial) geodesic  $H_t$  starting at  $H_0$  with eigenvalues  $(\mu_i^{(k)})$ . Then our normalization condition implies that  $\mu_{\max} := \max_i (\mu_i^{(k)})$  is positive. Thus, computing in the similar way as in [Don09, Proposition 3], we have

$$(\epsilon_k J^{(k)} - \mathcal{D}_{g_{V_{KS}}}^{(k)} + \mathcal{D}_{g_{V_k}}^{(k)})(H_t) = \epsilon_k \mathcal{L}_{\mu_0}(FS_k(H_t)) \geq \epsilon_k \mu_{\max} t + (\text{const}) \rightarrow \infty$$

as  $t \rightarrow \infty$ . Hence the functional  $\epsilon_k J^{(k)} - \mathcal{D}_{g_{V_{KS}}}^{(k)} + \mathcal{D}_{g_{V_k}}^{(k)}$  is coercive. To get the second assertion  $\epsilon'_k \rightarrow 0$ , we use the  $g$ -analogue of calculation techniques developed in [BBGZ13, Section 7]. In what follows all  $O(1)$  and  $o(1)$  are meant to hold uniformly with respect to  $H \in \mathcal{H}_k^{T_c}$  as  $k \rightarrow \infty$ . We use the normalization

$$\mathcal{L}_{\mu_0}(FS_k(H)) = 0$$

so that

$$(3.15) \quad \sup_X (FS_k(H) - \phi_0) \leq O(1),$$

and a reference point  $\tilde{H}_0 := \text{Hilb}_{k,\mu_0,\epsilon_k+g_{V_{KS}}-g_{V_k}}(\phi_0)$ . Let  $H_t$  be a geodesic joining  $\tilde{H}_0$  to  $H := H_1 \in \mathcal{H}_k^{T_c}$  and put

$$v(H) := \left. \frac{\partial}{\partial t} \right|_{t=0} FS_k(H_t).$$

Then we have the formula

$$(3.16) \quad \mathcal{E}_{\epsilon_k+g_{V_{KS}}-g_{V_k}}^{(k)}(H) = \int_X v(H) \beta_{k,\mu_0,\epsilon_k+g_{V_{KS}}-g_{V_k}}(\phi_0) + o(1),$$

where the error term  $o(1)$  in the RHS comes from the change of base points from  $H_0$  to  $\tilde{H}_0$ . Then  $v(H)$  is estimated as

$$\begin{aligned} v(H) &\leq FS_k(H) - FS_k(\tilde{H}_0) \quad (\text{by the convexity of } FS_k(H_t)) \\ &= FS_k(H) - FS_k(\text{Hilb}_{k,\mu_0,\epsilon_k+g_{V_{KS}}-g_{V_k}}(\phi_0)) \\ &= FS_k(H) - \phi_0 - \frac{1}{k} \log \left( \frac{1}{N_k} \rho_{k,\mu_0,\epsilon_k+g_{V_{KS}}-g_{V_k}}(\phi_0) \right) \\ &\leq FS_k(H) - \phi_0 - \frac{1}{k} \log \left( \frac{1}{N_k} 2^{-k} \rho_{k,\mu_0}(\phi_0) \right) \\ &\quad (\text{by the definition of } \epsilon_k) \\ &\leq O(1) \quad (\text{by (3.15) and Proposition 3.2}). \end{aligned}$$

On the other hand, by the uniform convergence  $g_{V_k} \rightarrow g_{V_{KS}}$  and Proposition 3.2, the positive measure  $\beta_{k,\mu_0,\epsilon_k+g_{V_{KS}}-g_{V_k}}(\phi_0)$  goes to 0 uniformly as  $k \rightarrow \infty$ . Hence we obtain

$$\mathcal{E}_{\epsilon_k+g_{V_{KS}}-g_{V_k}}^{(k)}(H) \leq \sup_X v(H) \int_X \beta_{k,\mu_0,\epsilon_k+g_{V_{KS}}-g_{V_k}}(\phi_0) + o(1) \leq \epsilon'_k$$

for some positive number  $\epsilon'_k \rightarrow 0$ . Therefore

$$(\epsilon_k J^{(k)} - \mathcal{D}_{g_{V_{KS}}}^{(k)} + \mathcal{D}_{g_{V_k}}^{(k)})(H) = -\mathcal{E}_{\epsilon_k+g_{V_{KS}}-g_{V_k}}^{(k)}(H) \geq -\epsilon'_k.$$

One can prove another inequality in the similar way.  $\square$

By Lemma 3.5, we have

$$\begin{aligned} \mathcal{D}_{g_{V_k}}^{(k)}(H) &= \mathcal{D}_{g_{V_k}}^{(k)}(F^*H) \quad (\text{Because } \text{Fut}_{V_k,k} \equiv 0) \\ &\geq \mathcal{D}_{g_{V_{KS}}}^{(k)}(F^*H) - \epsilon_k J^{(k)}(F^*H) - \epsilon'_k \\ &\geq \left( \frac{\delta''}{2} - \epsilon_k \right) J^{(k)}(F^*H) - 2C - \epsilon'_k. \end{aligned}$$

Thus we have

$$(3.17) \quad \mathcal{D}_{g_{V_k}}^{(k)}(H) \geq \frac{\delta''}{3} \inf_{F \in \text{Aut}_0(X, V_{KS})} J^{(k)}(F^*H) - 3C$$

for sufficiently large  $k$ . Since  $J^{(k)}$  is an exhaustion function on  $\mathcal{H}_k^{T_c}/\mathbb{R}$ , we find that there exists a unique quantized Kähler-Ricci soliton  $H_k$  at level  $k$  up to the action of  $\text{Aut}_0(X, V_{KS})$  if  $k$  is sufficiently large. We normalize  $H_k$  so that the corresponding

metric  $\phi_k := FS_k(H_k)$  minimizes  $J$  on the corresponding  $\text{Aut}_0(X, V_{KS})$ -orbit. Then the minimizing property of  $H_k$  implies  $\mathcal{D}_{g_{V_k}}^{(k)}(H_k) \leq \mathcal{D}_{g_{V_k}}^{(k)}(\text{Hilb}_{k, \mu_0}(\phi))$  for all  $\phi \in \mathcal{H}(X, -K_X)^{T^c}$ . Thus letting  $k \rightarrow \infty$ , we obtain

$$(3.18) \quad \mathcal{D}_{g_{V_k}}^{(k)}(H_k) \leq \mathcal{D}_{g_{V_{KS}}}(\phi) + \gamma_k$$

for all  $\phi \in \mathcal{H}(X, -K_X)^{T^c}$ , where  $\gamma_k = \gamma_k(\phi) \rightarrow 0$  is a sequence of constants depending on  $\phi$ . On the other hand, we have

$$\begin{aligned} \mathcal{D}_{g_{V_{KS}}}(\phi_k) &\leq \mathcal{D}_{g_{V_{KS}}}^{(k)}(H_k) + \delta_k J^{(k)}(H_k) + \delta_k \quad (\text{Lemma 3.4 and } g_{V_{KS}} \text{ is bounded}) \\ &\leq \mathcal{D}_{g_{V_k}}^{(k)}(H_k) + \delta'_k J^{(k)}(H_k) + \delta'_k \quad (\text{by Lemma 3.5}), \end{aligned}$$

where  $J^{(k)}(H_k)$  is bounded from above by (3.17) and (3.18). Thus we have

$$\liminf_{k \rightarrow \infty} \mathcal{D}_{g_{V_{KS}}}(\phi_k) \leq \mathcal{D}_{g_{V_{KS}}}(\phi)$$

for all  $\phi \in \mathcal{H}(X, -K_X)^{T^c}$ . Since the set  $\mathcal{H}(X, -K_X)^{T^c}$  contains a Kähler-Ricci soliton with respect to  $V_{KS}$ , i.e., a minimizer of  $\mathcal{D}_{g_{V_{KS}}}$  on  $\mathcal{E}^1(X, -K_X)^{T_{KS}}$ , we have

$$\liminf_{k \rightarrow \infty} \mathcal{D}_{g_{V_{KS}}}(\phi_k) \leq \inf_{\phi \in \mathcal{E}^1(X, -K_X)^{T_{KS}}} \mathcal{D}_{g_{V_{KS}}}(\phi).$$

This yields that  $\{\phi_k\}$  is a minimizing sequence of the functional  $\mathcal{D}_{g_{V_{KS}}}$ . Since  $J^{(k)}(H_k)$  is bounded,  $J(\phi_k)$  is also bounded by Lemma 3.4. Thus  $\{\phi_k\}$  is contained in a compact sublevel set of  $J$ , and there exists a subsequence which converges to some metric  $\phi_\infty \in \mathcal{E}^1(X, -K_X)^{T_{KS}}$ . Since  $\mathcal{D}_{g_{V_{KS}}}$  is lower semi-continuous (cf: [BBGZ13, Lemma 6.4], [BN14, Proposition 2.15]), we obtain

$$\mathcal{D}_{g_{V_{KS}}}(\phi_\infty) \leq \liminf_{k \rightarrow \infty} \mathcal{D}_{g_{V_{KS}}}(\phi_k) \leq \inf_{\phi \in \mathcal{E}^1(X, -K_X)^{T_{KS}}} \mathcal{D}_{g_{V_{KS}}}(\phi),$$

hence  $\phi_\infty$  is a Kähler-Ricci soliton with respect to  $V_{KS}$ , which is smooth by the regularity theorem [BN14, Theorem 1.3]. The metric  $\phi_\infty$  may depend on a choice of a convergent subsequence. However, by our normalization of  $\phi_k$ , we know that  $\phi_\infty$  minimizes  $J$ -functional on the space of Kähler-Ricci solitons with respect to  $V_{KS}$ , which can be identified with the space  $\text{Aut}_0(X, V_{KS})\phi_\infty/K$ , where  $K$  is the stabilizer of  $\phi_\infty$  (cf. [BN14, Theorem 3.6]). Since  $J$  is strictly convex on  $\text{Aut}_0(X, V_{KS})\phi_\infty/K$  with respect to the natural Riemannian structure (where geodesics are one parameter subgroups), such a minimizer is uniquely determined. Therefore, the metric  $\phi_\infty$  is, in fact, independent of a choice of a subsequence, which yields that  $\phi_k$  converges to a Kähler-Ricci soliton  $\phi_\infty$  weakly. This completes the proof.  $\square$

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